ON EXPLICIT FORM OF EXTENSION OPERATOR FOR C^∞- FUNCTIONS

A.P.GONCHAROV

Department of Mathematics &
Bilkent University
06533 Ankara, Turkey

Department of Mathematics
Civil Building University
Rostov-na-Donu, Russia

ABSTRACT. We give an explicit form of a continuous linear extension operator $Q : \mathcal{E}(K) \to C^\infty(\mathbb{R}^d)$ by extending the elements of basis of the space $\mathcal{E}(K)$ for model compact sets K being the closure of a plane domain with a cusp and its one-dimensional analog: a sequence of closed intervals tending to a point. The method generalizes Mitiagin's construction ([9]) of the extension operator for $K = [-1, 1] \subset \mathbb{R}$.

1991 *Mathematics Subject Classification.* Primary 26E10, 46A35, 41A10 ; Secondary 46E10, 41A17.

Key words and phrases. linear extension operator, bases, Markov's inequality.

Explicit form of extension operator

Department of Mathematics
Bilkent University
06533 Ankara, Turkey
E-mail: goncha@fen.bilkent.edu.tr

1. **Introduction.**

Let X be a closed subset of $\mathbb{R}^d$ and $m \in \mathbb{N}$. Whitney's extension theorem [20] gives an extension operator (here and in what follows it means a continuous linear extension operator) from the space $\mathcal{E}^m(X)$ of Whitney jets on X to the space $C^m(\mathbb{R}^d)$. In the case $m = \infty$ such an operator does not exist in general and several authors have considered the extension problem in different situations. Mitiagin [9] presented an extension operator for a closed interval in $\mathbb{R}$, Seeley [15] did so for a half-space of $\mathbb{R}^d$. Applying Whitney's method, Stein [16] constructed an extension operator when X is the closure of a Lipschitz domain in $\mathbb{R}^d$. Bierstone [1] proved the existence of an extension operator for X with the boundary of Hölder's type, Tidten [17] did so for closed subsets of $\mathbb{R}^d$ admitting some polynomial cusps. In [11] (see also [12]) Pawłucki and Pleśniak gave a construction of a such operator for compact sets satisfying the Markov Property (see Section 5). In elaboration of Whitney's method Schmets and Valdivia proved in [14] that the existence of an extension operator $Q : \mathcal{E}(K) \to C^\infty(\mathbb{R}^d)$ for a compact set $K \subset \mathbb{R}^d$ implies the possibility to take such a map for which all extensions are analytic on $\mathbb{R}^d \setminus K$. For the extension problem in the classes of ultradifferentiable functions see for instance [2] and [13] and the references given there.

In this paper we construct an extension operator $Q : \mathcal{E}(K) \to C^\infty(\mathbb{R}^d)$ by extending the elements of basis of the space $\mathcal{E}(K)$ for some model cases of a compact set K. The idea to give an extension operator in this manner goes back to Mitiagin [9].

2. **Preliminaries.**

Let $K \subset \mathbb{R}^d$ be a compact set. Suppose that $K = \overline{Int(K)}$. The space $\mathcal{E}(K)$ of Whitney functions on K is the space of functions $f : K \to \mathbb{R}^d$ extendable to C^∞- functions on $\mathbb{R}^d$. The topology of Fréchet space in $\mathcal{E}(K)$ is given by the norms

$$\|f\|_p = |f|_p + \sup\left\{ \frac{|(R^p_{z_0}f)^{(j)}(z)|}{|z - z_0|^{p-|j|}} : \ z, z_0 \in K, \, z \neq z_0, \, |j| \leq p \right\},$$

$p \in \mathbb{N}_0 := \{0, 1, ...\}$, where $|f|_p = \sup\{|f^{(j)}(z)| : z \in K, |j| \leq p\}$, $j = (j_1, ..., j_d) \in \mathbb{N}_0^d$ with $|j| = j_1 + ... + j_d$ and $R^p_{z_0}f(z) = f(z) - T^p_{z_0}f(z)$ is the Taylor remainder. In what follows we will consider only the cases $d = 1$ or $d = 2$ and the following model compact sets:

1) $K = \{0\} \cup \bigcup_{k=1}^\infty I_k \subset [0, 1] \subset \mathbb{R}$, where $I_k = [a_k, b_k] = [x_k - \delta_k, x_k + \delta_k]$ with $a_k \downarrow 0$ and $h_k := a_k - b_{k+1} > 0$ for all k.

2) $K_\psi = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ |y| \leq \psi(x)\}$, where ψ is a nondecreasing function on $[0, 1]$, $0 \leq \psi(x) \leq x$, $\psi(+0) > 0$.

We use the Chebyshev polynomials

$$T_n(x) = \cos(n \cdot \arccos x), \, |x| \leq 1, \, n \in \mathbb{N}_0.$$

Let $\tilde{T}_n$ be the Chebyshev polynomial considered on $\mathbb{R}$ and for fixed interval I_k let $\tilde{T}_{nk}$ denote the scaling Chebyshev polynomial, that is $\tilde{T}_{nk}(x) = \tilde{T}_n(\frac{x-x_k}{\delta_k})$, and let T_{nk} be the restriction of $\tilde{T}_{nk}$ on I_k, $T_{nk} = 0$ otherwise on K.

By ξ_{nk} we denote the functional $\xi_{nk}(f) = \frac{2}{\pi}\int_0^\pi f(x_k + \delta_k \cos t)\cos ntdt$, $n \in \mathbb{N}_0$ (if $n = 0$, then we take 1 instead of 2 in the coefficient). Clearly, for fixed k the functionals (ξ_{nk}) are biorthogonal to the system $(\tilde{T}_{nk})$.

By $|\cdot|_{-q}$ we denote the dual norm of a functional in the corresponding space. We adhere to the convention that $\sum_{i=m}^{n} = 0$ for $m > n$ and $0^0 = 1$, log will denote the natural logarithm.

3. **Basis for the first model case.**

Let us give a generalization of the basis construction from [7]. Suppose that for some constant C_0

$$b_k \le C_0\delta_k, \ \ \delta_k \le C_0 h_k, \ k \in \mathbb{N}. \tag{1}$$

Let $l : \mathbb{N} \to \mathbb{N}_0$ be a nondecreasing function. For fixed $k \in \mathbb{N}$ and $n < l(k)$ let $e_{nk} = \tilde{T}_{nk}\Big|_{[0,b_k]\cap K}$ and $e_{nk} = 0$ otherwise on K. If $n \ge l(k)$, then let $e_{nk} = T_{nk}$. To introduce the biorthogonal functionals we take $\eta_{nk} = \xi_{nk}$ for $n \ge l(k)$. If $n < l(k)$, then η_{nk} be the projection of ξ_{nk} on the subspace spanned by the previous functionals, that is

$$\eta_{nk} = \xi_{nk} - \sum_i \xi_{nk}(e_{i\,k-1})\,\xi_{i\,k-1}.$$

Since ξ_{nk} is biorthogonal to all polynomials of degree less than n and to all $e_{i\,k-1}$ which are $T_{i\,k-1}$ for $i \ge l(k-1)$, the sum above is only over $i = n, \cdots, l(k-1) - 1$. The system of functionals $(\eta_{nk})_{n=0,k=1}^{\infty,\infty}$ is total on $\mathcal{E}(K)$ and biorthogonal to $(e_{nk})_{n=0,k=1}^{\infty,\infty}$ (see [7], L.3.2). In order to use the Dynin-Mitiagin criterion of property of being a basis ([9],T.9):

$$\forall p \exists q, C: \ \ \|e_{nk}\|_p \cdot |\eta_{nk}|_{-q} \le C, \ \ \forall n, k, \tag{2}$$

we choose the function l in the following way. Since the sequence $(\delta_k)_{k=1}^\infty$ is not monotone in general, let $\tilde{\delta}_k = max\{\delta_j, j \ge k\}$. Then $\tilde{\delta}_k \downarrow 0$ and $\tilde{\delta}_k \ge \delta_k$. Let $l(k) = [\log \tilde{\delta}_{k-1}^{-1} / \log(3C_0)]$, where $[a]$ denotes the greatest integer in a. Then $l \uparrow \infty$ and

$$l(k) \le \delta_{k-1}^{-1}, \ \ (3C_0)^{l(k)} \le \delta_{k-1}^{-1}, \ \ k \ge 2. \tag{3}$$

Theorem 1. *Let for a compact set $K = \{0\} \cup \bigcup_{k=1}^\infty I_k$ the assumption (1) hold. Then the system $\{e_{nk}, \eta_{nk}\}_{n=0,k=1}^{\infty,\infty}$ is a basis in the space $\mathcal{E}(K)$.*

Proof: Fix $p \in \mathbb{N}_0$. Let $q = 3p+3$. To simplify notation here and in what follows we use the same letter C for any coefficient which does not depend on n and k.

We first obtain some estimations for the case $n < l(k), k \in \mathbb{N}$. For $n < p$ L.4.2 in [7] and (1) imply

$$\|e_{nk}\|_p \le C 2^p p!\, \delta_k^{-n} h_{k-1}^{-p} \le C \delta_k^{-n} \delta_{k-1}^{-p}.$$

If $p \le n < l(k)$, then similarly

$$\|e_{nk}\|_p \le C 2^n n^p\, \delta_k^{-n} b_k^{n-p} h_{k-1}^{-p} \le C 2^l l^p \delta_k^{-p} \delta_{k-1}^{-p}.$$

Thus in both cases, applying (3) we see that

$$\|e_{nk}\|_p \le C \delta_k^{-\min(n,p)} \delta_{k-1}^{-2p-1}. \tag{4}$$

On the other hand, for functionals we get as in [7]

$$\sum_{i=n}^{l(k-1)-1} |\xi_{nk}(e_{i\,k-1})| \le \sum_{i=n}^{l-1} (\frac{\delta_k}{\delta_{k-1}})^n (2\frac{x_{k-1}-x_k}{\delta_{k-1}} + 3)^i < (\frac{\delta_k}{\delta_{k-1}})^n\, (3C_0)^{l(k)}$$

$< (\frac{\delta_k}{\delta_{k-1}})^n\, \delta_{k-1}^{-1}$, by (3). Therefore, by L.4.1 in [7]

$$|\eta_{nk}|_{-q} \le C\, [\delta_k^q + \delta_{k-1}^{q-1} (\delta_k/\delta_{k-1})^n].$$

Besides, $(\frac{\delta_k}{\delta_{k-1}})^n \le \delta_k^{\min(n,p)} \delta_{k-1}^{-p-1}$. In fact, for $n \le p$ it is trivial; for $n > p$ (1) implies that $\delta_k < C_0 \delta_{k-1}$ and $(\frac{\delta_k}{\delta_{k-1}})^{n-p} \le C_0^l < \delta_{k-1}^{-1}$, by (3). Thus,

$$|\eta_{nk}|_{-q} \le C\, [\delta_k^q + \delta_k^{\min(n,p)} \delta_{k-1}^{q-p-2}]. \tag{5}$$

Now for given $l(k)$ and q let us fix k_q such that $l(k) \ge q$. We decompose $\mathbb{N}_0 \times \mathbb{N}$ into three disjoint zones: $Z_0 = \{(n,k) : 1 \le k \le k_q,\, 0 \le n < m := \max(q, l(k))\}$, $Z_1 = \{(n,k) : 1 \le k,\, m \le n\}$, $Z_2 = \{(n,k) : k_q < k,\, n < l(k)\}$.

The zone Z_0 contains only finite number of elements, hence the products $\|e_{nk}\|_p \cdot |\eta_{nk}|_{-q}$ are uniformly bounded here.

If $(n,k) \in Z_1$, then $n \ge l(k)$. Here as in T.5.1 from [7] we have $\|T_{nk}\|_p \le C n^{2p} \delta_k^{-p}$ and $|\xi_{nk}|_{-q} \le C\, (\delta_k/n)^q$. Therefore, the products $\|T_{nk}\|_p \cdot |\xi_{nk}|_{-q}$ are uniformly bounded as well.

The same conclusion can be drawn for the zone Z_2 by (4) and (5) due to the choice of q. This gives (2), and the proof is complete. □

4. **Continuous linear extension operator .**

For compact sets from the previous section (under the assumption of monotonicity of $(\delta_k), (h_k)$) we have the following geometric criterion of the extension property (see [6], T.3): an extension operator exists if and only if for some constant M and for all k

$$\delta_k \ge \delta_{k-1}^M. \tag{6}$$

Let us show that whenever this operator exists (for compact sets with (1)) it can be given by extending of the basis elements of the space $\mathcal{E}(K)$.

Given an interval $[a,b]$ and $\tau > 0$ let $\omega = \omega(a,b,\tau,x)$ be a $C^\infty-$function with the following properties: $\omega(x) = 1$ for $x \in [a,b]$; $\omega(x) = 0$ if $dist(x,[a,b]) \ge \tau$ and $|\omega^{(j)}|_0 \le C_j \tau^{-j}$ for some constant C_j, $j \in \mathbb{N}_0$.

Set $\tau_{nk} = C_0^{-2}(n^2+1)^{-1}\delta_k$ for $(n,k) \in \mathbb{N}_0 \times \mathbb{N}$. Then $\min\{h_k, h_{k-1}\} \geq \tau_{nk}, \forall n, k$, as is easy to see. Let $\omega_{nk}(x) = \omega(0, b_k, \tau_{nk}, x)$ for $n < l(k)$ and $\omega_{nk}(x) = \omega(a_k, b_k, \tau_{nk}, x)$ for $n \geq l(k)$. Clearly,

$$|\omega_{nk}^{(j)}|_0 \leq C(n^2+1)^j\, \delta_k^{-j},\ j \in \mathbb{N}_0, (n,k) \in \mathbb{N}_0 \times \mathbb{N}. \tag{7}$$

Define $\tilde{e}_{nk} = \tilde{T}_{nk} \cdot \omega_{nk}$ and

$$Q : \mathcal{E}(K) \to C^\infty(\mathbb{R}) : f \mapsto \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \eta_{nk}(f) \cdot \tilde{e}_{nk}.$$

Theorem 2. *Let for a compact set $K = \{0\} \cup \bigcup_{k=1}^\infty I_k$ the assumptions (1) and (6) hold. Then Q is a continuous linear extension operator.*

Proof: Since $\tilde{e}_{nk}|_K = e_{nk}$ due to the choice of τ_{nk}, we see that Q is an extension operator. Clearly it is linear. Let us show that Q is well-defined and continuous. Given $p \in \mathbb{N}_0$ let $q = (M+3)p+4$. Let the function $l(k)$ and k_q be the same as in the previous section.

For each polynomial P the extremal properties of Chebyshev's polynomials imply the following bound

$$|P(x)| \leq |x + \sqrt{x^2-1}|^{degP} \sup\{|P(x)| : |x| \leq 1\},\ \ |x| > 1.$$

Therefore we get $|\tilde{T}_n^{(i)}(1+\varepsilon)| \leq (1+2\sqrt{\varepsilon})^n T_n^{(i)}(1)$ if $\varepsilon \leq 1/4$ and $|\tilde{T}_n^{(i)}(x)| \leq e^2(n^2+1)^i$ for $|x| \leq 1 + (n^2+1)^{-1}$. It follows that if $dist(x, I_k) \leq \tau_{nk}$, then $|\tilde{T}_{nk}^{(i)}(x)| \leq e^2(n^2+1)^i \delta_k^{-i}$.

Using the Leibnitz formula and (7) we get for $n \geq l(k)$

$$|\tilde{e}_{nk}|_p \leq C(n^2+1)^p \delta_k^{-p}. \tag{8}$$

Consider $0 < n < l(k)$ and x with $dist(x, [0, b_k]) \leq \tau_{nk}$. The polynomial $\tilde{T}_{nk}$ can be written in the form

$$\tilde{T}_{nk}(x) = 2^{n-1}\delta_k^{-n} \prod_{j=1}^{n}(x - \theta_j), \tag{9}$$

where $\theta_j \in I_k$. Since $|x - \theta_j| < b_k + \tau_{nk} < (C_0+1)\delta_k$, an easy computation shows that

$$|\tilde{T}_{nk}^{(i)}(x)| \leq 2^{n-1}\delta_k^{-n} n^{-i}[(C_0+1)\delta_k]^{n-i} < \delta_k^{-i}\delta_{k-1}^{-i-1},$$

by (3). From this as before

$$|\tilde{e}_{nk}|_p \leq C\delta_k^{-p}\delta_{k-1}^{-2p-1}. \tag{10}$$

Clearly it is valid also for $n = 0$.

Fix $f \in \mathcal{E}(K)$. To deal with $|Q(f)|_p$, we use the following decomposition corresponding to the chosen zones

$$|Q(f)|_p \leq \left(\sum_{k=1}^{k_q}\sum_{n=0}^{m-1} + \sum_{k=1}^{\infty}\sum_{n=m}^{\infty} + \sum_{k=k_q+1}^{\infty}\sum_{n=0}^{l(k)-1}\right) |\eta_{nk}(f)| \cdot |\tilde{e}_{nk}|_p.$$

Let us consider the double sums above separately. The first sum contains only a finite number of items, hence it is bounded from above by $C\|f\|_q$, where the constant C does not depend on n and k.

For the terms of the second sum we have as before

$$|\eta_{nk}(f)| = |\xi_{nk}(f)| \le C\,(\delta_k/n)^q \|f\|_q,$$

which gives the desired conclusion when combined with (8).

For the last sum we can rewrite (5) in the form

$$|\eta_{nk}(f)| \le C\,(\delta_k^q + \delta_{k-1}^{q-p-2})\|f\|_q.$$

The number of summands with respect to n here is $l(k)$, which is smaller than δ_{k-1}^{-1}. Taking into account (10) and (6), we see that the last series converges as well.

Thus the operator Q is well defined and $|Q(f)|_p \le C\|f\|_q$.

□

Remark. The case of a compact set K with the property $\delta_k = o(\delta_{k-1}^M), \forall M$, corresponds to a plane domain with the sharp cusp. The basis in the space $\mathcal{E}(K)$ can be constructed here as well, but the extension operator does not exist. Analytically speaking, there are no τ_{nk} suitable for all n, k from the zone Z_2.

5. **Comparing two methods of extension .**

In [11] (see also [12]) Pawłucki and Pleśniak suggested an extension operator $Q : \mathcal{E}(K) \to C^\infty(\mathbb{R}^d)$ in the form of a series containing Lagrange interpolation polynomials with Fekete-Leja system of knots. The basic assumption for their construction was the following Markov Property of a compact set K:

$$\exists C, \mu : \ \ |P^{(j)}|_0 \le C \cdot (degP)^{\mu|j|}|P|_0, \ \forall j \in \mathbb{N}_0^d, \ \forall P.$$

Here P is a polynomial, $|\cdot|_0$ is considered in the space $\mathcal{E}(K)$.

Our method of extension has the disadvantage of being very special. At the same time it is "more explicit", since the disposition of Fekete-Leja system of extremal points is only known for a few types of compact sets. Besides it can be applied for some classes of compact sets without Markov's Property.

Consider, as an example, the case

$$\delta_{k+1} = \delta_k^M, \ \ b_k = C_0\delta_k, \ k \in \mathbb{N}, \qquad (11)$$

with $M \ge 2, \ C_0 \ge 6$. Then the hypothesis of Theorem 2 is fulfilled but the Markov inequality is not satisfied for certain polynomials on K. (Compare this with [5].)

Proposition 1. *The compact set K under the assumption (11) does not have the Markov property.*

Proof: Without loss of generality let $\delta_k = \exp(-M^k)$, $k \in \mathbb{N}$. Fix $m \in \mathbb{N}$ and consider the polynomial $P(x) = x \cdot \prod_{k=1}^m \gamma_k \cdot \tilde{T}_{n_k k}(x)$, where $\gamma_k = \tilde{T}_{n_k k}^{-1}(0)$. We take $n_m = 1$, $n_k = M^{m+(m-1)+\cdots+(k+1)}$ for $k \le m-1$.

Clearly, $P'(0) = 1$ and $degP = 1 + \sum_{k=1}^m n_k < M^{m^2}$. We will show that

$$|P(x)| \le b_m, \ x \in K.$$

It will follow the absence of the Markov property of K as

$$1 \le CM^{\mu m^2} C_0 \exp(-M^m), \quad m \to \infty$$

is a contradiction for fixed C, μ.

Fix $x \in K$. If $x \le b_m$, then $|\gamma_k \cdot \tilde{T}_{n_k k}(x)| \le 1$, $k = 1, 2, ..., m$, and the desired bound of $|P(x)|$ is obvious. Consider now $x \in I_j$, $1 \le j \le m-1$. Then

$$|P(x)| \le b_j |\gamma_j| \prod_{k=j+1}^{m} |\gamma_k \cdot \tilde{T}_{n_k k}(x)|,$$

as all other terms of the product are less than 1.

To estimate the remaining terms, we use the bound

$$2^{n-1}(\Delta_k/\delta_k)^n < |\tilde{T}_{nk}(x)| < 2^{n-1}(\Delta_k/\delta_k + 2)^n, \ n > 0, \ \Delta_k = dist(x, I_k),$$

which is clear from (9).

Therefore, $|\gamma_k \cdot \tilde{T}_{n_k k}(x)| \le (\frac{b_j}{a_k})^{n_k} = (\frac{C_0 \delta_j}{(C_0-2)\delta_k})^{n_k}$ and $|\gamma_j| < 2(2C_0-4)^{-n_j}$.

Hence,

$$|P(x)| < 2C_0 \exp(-M^j)(2C_0-4)^{-n_j} \exp \sum_{k=j+1}^{m} n_k [M^k - M^j + \log(\frac{C_0}{C_0-2})].$$

Since $M^j \ge M > \log \frac{3}{2} \ge \log(\frac{C_0}{C_0-2})$ due to the choice of M, C_0; $2\exp(-M^j) < 1$ and $n_k M^k = n_{k-1}$, we have

$$\log(|P(x)|/b_m) < M^m - n_j \log(2C_0-4) + \sum_{k=j}^{m-1} n_k.$$

From $M^m + \sum_{k=j}^{m-1} n_k \le 2n_j$, $\log(2C_0-4) > 2$ it follows that the expression on the right is negative and $|P|_0 \le b_m$, as claimed. $\Box$

6. Bases and extension operators for the space $C^\infty(\bar{\Omega}_\psi)$.

We now turn to the case of the compact set K_ψ being the closure of the plane domain $\Omega_\psi = IntK_\psi$ of the cusp form. Since the set K_ψ is regular in Whitney sense, we have $C^\infty(\bar{\Omega}_\psi) \simeq \mathcal{E}(\bar{\Omega}_\psi)$, where $C^\infty(\bar{\Omega})$ is the space of infinitely differentiable in Ω functions such that the functions and all their derivatives are uniformly continuous on the domain, equipped with the norms $(|\cdot|_p)_{p=0}^\infty$.

To analyze topological properties of the space $C^\infty(\bar{\Omega})$ the property $\bar{\Omega}$ of being uniformly polynomially cuspidal (see [10], [11]) is important. For our case it can be given by the following condition:

$$(12) \qquad \exists N,\, \tau_0: \;\; \psi(\tau) \geq \tau^N, \;\; 0 \leq \tau \leq \tau_0.$$

(Without loss of generality we don't allow the domain Ω_ψ to have a cusp at the points $(1, \pm\psi(1))$.)

Due to the Vogt-Tidten criterion ([17],[19],T.2.4) an extension operator $Q : \mathcal{E}(K) \to C^\infty(\mathbb{R}^d)$ exists for a compact set K with $Int(K) \neq \emptyset$ if and only if the space $\mathcal{E}(K)$ is isomorphic to the space s of rapidly decreasing sequences. In particular Tidten [17] (see also [1]) proved the existence of a such operator for compact sets admitting polynomial cusps.

On the other hand, for the domain Ω_ψ we have the following characterization([4],T.1.3):

Theorem 3. *The following statements are equivalent:*
(i) the function ψ satisfies the condition (12);
(ii) the compact set $\bar{\Omega}_\psi$ has Markov's property;
(iii) $C^\infty(\bar{\Omega}_\psi) \simeq s$.

For the convenience of the reader we briefly sketch the proofs.

The implication $(i) \Rightarrow (ii)$ can be obtained by using the Hölder continuity property of the Green function with the pole at infinity for the domain $\mathbb{C}\setminus\bar{\Omega}_\psi$ (see, e.g. [10], where Pawłucki and Pleśniak proved the Markov property for wide class of uniformly cuspidal subsets in $\mathbb{R}^d$.)

For $(ii) \Rightarrow (iii)$ we can use [3], where a basis was constructed in the space $C^\infty(\bar{\Omega})$ for the domain $\Omega \subset \mathbb{R}^d$ with the boundary of Hölder's type (see also [22] for a more general case). The basis can be constructed out of the polynomials $(P_n)_{n=0}^\infty$ ortogonalized in the Sobolev space $W_2^{(r)}(\Omega)$ with certain natural r depending on the domain. In our case one can take $r > \frac{\mu+2}{4}$ with μ being fixed from the definition of the Markov property. Then for any function from the space $C^\infty(\bar{\Omega}_\psi)$ the sequence of coefficients of its basis expansion rapidly decreases.

To prove $(iii) \Rightarrow (i)$ we use the fact that the space $C^\infty(\bar{\Omega}_\psi)$ belongs to the class D_1 ([21]) or has the dominating norm property DN ([18]) as the space which is isomorphic to s, that is $\exists p : \forall q \;\exists r, C$:

$$(13) \qquad |f|_q^2 \leq C|f|_p|f|_r, \quad f \in C^\infty(\bar{\Omega}_\psi),$$

where $p, q, r \in \mathbb{N}_0$, $C > 0$.

Suppose, contrary to our claim, that for some sequence $(\tau_n), \tau_n \downarrow 0$ we have

$$(14) \qquad \psi(\tau_n) < \tau_n^n, \;\; n \in \mathbb{N}.$$

For the function $\omega_n(x) = \omega(0, \tau_n/2, \tau_n/2, x)$ let us take $f_n(x,y) = y^{p+1}\omega_n(x), (x,y) \in \Omega_\psi, n \in \mathbb{N}$, where p is fixed from the definition above. Set $q = p+1$ and fix r, C such that (13) holds. Using (14), it is easy to check that $|f|_p \leq C_p\psi(\tau_n) < C_p\tau_n^n$, $|f|_q \geq |f_{y^q}^{(q)}(0,0)| \geq (p+1)!$, $|f|_r \leq C_r\tau_n^{p+1-r}$, where the constants C_p, C_r do not depend on n.

Substituting these bounds into (13) we get a contradiction for big n. Hence, (14) is impossible.

Thus, whenever an extension operator $Q : C^\infty(\bar{\Omega}_\psi) \to C^\infty(\mathbb{R}^2)$ exists, the construction of Pawłucki and Pleśniak gives an explicit form of Q. At the same time it can be given by extending the basis elements $(P_n)_{n=0}^\infty$ of the space $C^\infty(\bar{\Omega}_\psi)$.

In fact, let $\tilde{\omega}(x,y) = \tilde{\omega}(\bar{\Omega}_\psi, \tau, x, y) \in C^\infty(\mathbb{R}^2)$ be a function such that $\tilde{\omega}(x,y) = 1$ for $(x,y) \in \bar{\Omega}_\psi$, $\tilde{\omega}(x,y) = 0$ if $dist((x,y), \bar{\Omega}_\psi) \geq \tau$ and $|\tilde{\omega}|_p \leq C_p \tau^{-p}$, $p \in \mathbb{N}_0$.

Following Pleśniak ([12], T.3.3), by Markov's Property of $\bar{\Omega}_\psi$ we have for some constants M, μ_0 and for every polynomial P with $degP > 0$ the bound $|P(x,y)| \leq M|P|_0$ if $dist((x,y), \bar{\Omega}_\psi) \leq (degP)^{-\mu_0}$. We can certainly assume that μ_0 is the same as μ in the definition of the Markov Property, since otherwise we replace the smaller value by the larger one.

After normalization of the basis polynomials $(P_n)_{n=0}^\infty$ we have $|P_n|_0 = 1, n \in \mathbb{N}_0$. Clearly, $degP_n \leq n$. Extending the polynomials analytically we take $\tilde{P}_n = P_n \tilde{\omega}_n$, where $\tilde{\omega}_n = \tilde{\omega}(\bar{\Omega}_\psi, (degP_n)^{-\mu}, x, y)$ for $n \geq 1$ and $\tilde{\omega}_0 = \tilde{\omega}(\bar{\Omega}_\psi, 1, x, y)$.

Using the Leibnitz rule and the Markov property of $\bar{\Omega}_\psi$ we get $|\tilde{P}_n|_p \leq D_p(1+degP_n)^{\mu p}$, where D_p does not depend on n. But in the basis expansion $f = \sum_{n=0}^\infty \xi_n(f) P_n$ the sequence $(\xi_n(f))$ is rapidly decreasing, therefore the operator

$$Q : C^\infty(\bar{\Omega}_\psi) \to C^\infty(\mathbb{R}^2) : f \mapsto \sum_{n=0}^\infty \xi_n(f) \cdot \tilde{P}_n$$

is continuius and the following proposition holds.

Proposition 2. *If there exists a continuous linear extension operator $Q : C^\infty(\bar{\Omega}_\psi) \to C^\infty(\mathbb{R}^2)$, then it can be given by replacing all basis elements in the basis expansion of a function by their extensions with tilde.*

7. **The case of graduated cusp.**

Fix the sequence $(a_k)_{k=1}^\infty$ with the properties: $a_k \downarrow 0$; $\exists C : \frac{a_k}{C} \leq a_{k+1} \leq (1 - \frac{1}{C})a_k, \forall k$. For any sequence $(\psi_k)_{k=1}^\infty$ with $\psi_1 \leq 1$, $\psi_k \downarrow 0$ consider the step function $\psi : \psi(x) = \psi_k$ if $a_k \leq x < a_{k-1}, k \in \mathbb{N}$ (here $a_0 = 1$) and the corresponding domain Ω_ψ in the form of graduated cusp. In [8] a basis was constructed in the space $C^\infty(\bar{\Omega}_\psi)$ for arbitrary sharpness of the cusp Ω_ψ. If the function ψ satisfies (12), that is $\exists N :$

$$\psi_k \geq a_k^N, \quad k \in \mathbb{N}, \tag{15}$$

then there exists an extension operator, which can be given by both methods considered before.

On the other hand, following [8] we can construct a special basis in the space $C^\infty(\bar{\Omega}_\psi)$. At first we can choose a sequence $(b_k)_{k=1}^\infty$ such that $b_k - a_k = 2\delta_k \downarrow 0$ and the condition (1) holds. Denote by R_k the rectangle

$[a_k, b_k] \times [-\psi_k, \psi_k]$, by R'_k the rectangle $[b_k, a_{k-1}] \times [-\psi_k, \psi_k]$.
Set $K = \{0\} \cup \bigcup_{k=1}^{\infty} R_k$.

Let $e_{nmk}(x,y) = e_{nk}(x) T_m(\frac{y}{\psi_k})\Big|_K$, $n, m \in \mathbb{N}_0$, $k \in \mathbb{N}$. For $f \in \mathcal{E}(K)$ let

$$\xi_{nmk}(f) = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x_k + \delta_k \cos t, \psi_k \cos \tau) \cos nt \cos m\tau dt d\tau$$

(here instead of 4 we take 1 if $n = m = 0$ or 2 if $nm = 0, n+m \neq 0$). Set $\eta_{nmk}(f) = \xi_{nmk}(f)$ for $n \geq l(k)$, where $l(k)$ is the same as in Section 3. If $n < l(k)$ then let

$$\eta_{nmk}(f) = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} [f(x_k + \delta_k \cos t, \psi_k \cos \tau) \cos nt -$$

$$f(x_{k-1} + \delta_{k-1} \cos t, \psi_k \cos \tau) \cdot \sum_{i=n}^{l(k-1)-1} \xi_{nk}(e_{i\,k-1}) \cos it] \cos m\tau dt d\tau.$$

Arguing as in [8], we see that the system $\{e_{nmk}, \eta_{nmk}\}_{n,m=0,k=1}^{\infty,\infty}$ is a basis in the space $\mathcal{E}(K)$. Moreover the result still holds if we drop the assumption (9) in [8]: $\psi_k \leq \delta_k^2, k \in \mathbb{N}$, which was suitable for the sharp cusp but is unnecessarily restrictive here.

The task now is to construct a basis in the space $C^\infty(\bar{\Omega}_\psi)$. Let $\tilde{e}_{nmk}(x,y) = \tilde{e}_{nk}(x) T_m(\frac{y}{\psi_k}), (x,y) \in \Omega_\psi$, where $\tilde{e}_{nk}$ is the same as in Section 4. The derivative $\tilde{e}_{nk}^{(j)}(x)$ has the same (up to a factor C_j) upper bound as $e_{nk}^{(j)}(x)$ due to the choice of the parameters τ_{nk} in the smooting functions ω_{nk}. Therefore the projection

$$S : C^\infty(\bar{\Omega}_\psi) \to C^\infty(\bar{\Omega}_\psi) : f \mapsto \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \eta_{nmk}(f|_K) \cdot \tilde{e}_{nmk}$$

is well defined and continuous.

In this way we have the representation $C^\infty(\bar{\Omega}_\psi) = X_1 \oplus X_0$ with $X_1 = S(C^\infty(\bar{\Omega}_\psi))$, $X_0 = \{f \in C^\infty(\bar{\Omega}_\psi) : suppf \subset \bigcup_{k=2}^{\infty} R'_k\} = (\oplus_{k=2}^{\infty} S_k(C^\infty(\bar{\Omega}_\psi)))_s$, where $S_k(f) = f - S(f)$ on R'_k and 0 otherwise on Ω_ψ. The functions $(\tilde{e}_{nmk})_{n,m=0,k=1}^{\infty,\infty}$ give a basis in the subspace X_1. For the basis in the subspace $S_k(C^\infty(\bar{\Omega}_\psi))$ we take $\tilde{h}_{nmk}(x,y) = \tilde{h}_{nk}(x) T_m(\frac{y}{\psi_k})$, where $\tilde{h}_{nk}(x) = h_n(\tan(\frac{\pi}{2} \frac{2x - b_k - a_{k-1}}{b_k - a_{k-1}}))$ for $b_k < x < a_{k-1}$, $\tilde{h}_{nk}(x) = 0$ otherwise on $[0, b_1]$ and h_n is a classical Hermite function. Here we have used Mitiagin's construction ([9], L.26) of the basis in the space $C_0^\infty[-1,1]$ of $C^\infty-$ functions vanishing at the endpoints of the interval $[-1,1]$(see also [8]).

Our last goal is to construct an extension operator using this special basis. Set $\hat{\omega}(y) = \omega(-\psi_k, \psi_k, (m^2+1)^{-1}\psi_k, y)$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0$.
Let $\hat{e}_{nmk}(x,y) = \tilde{e}_{nk}(x) \tilde{T}_m(\frac{y}{\psi_k}) \hat{\omega}(y)$ and $\hat{h}_{nmk}(x,y) = \tilde{h}_{nk}(x) \tilde{T}_m(\frac{y}{\psi_k}) \hat{\omega}(y)$. Now the functions with hat belong to the space $C^\infty(\mathbb{R}^2)$. Since the proof of continuity of the corresponding extension operator is quite similar to the above, the details are left to the reader. Note that for the estimation

$|Q(f)|_p \leq C|f|_q$ we can take $q(p) \geq 3 + p \cdot \max\{N, 3\}$, where N is given in (15).

Proposition 3. *Let a graduated cusp domain* Ω_ψ *be defined by the sequence* (ψ_k) *satisfying (15). Then the functions* $\tilde{e}_{nmk}, \tilde{h}_{nmk+1}$, $n, m \in \mathbb{N}_0$, $k \in \mathbb{N}$ *form a special basis in the space* $C^\infty(\bar{\Omega}_\psi)$. *If in the basis expansion of a function we replace all basis elements with tilde by their extensions with hat, then the received map is a continuous linear extension operator* $C^\infty(\bar{\Omega}_\psi) \to C^\infty(\mathbb{R}^2)$.

References

1 E. Bierstone, Extension of Whitney-Fields from Subanalytic Sets, *Invent. Math* **46** (1978), 277-300.

2 J.Bonet, R.W.Braun, R.Meise, B.A.Taylor, Whitney extension theorem for nonquasianalytic classes of ultradifferentiable functions, of Roumieu type, *Studia Math.* **99**, (1991), 155-184.

3 A.P.Goncharov, An isomorphism of spaces of infinitely differentiable functions, *Izv. Sev. Kavkaz. Nauch. Centra Vysh. Skoly.*, **28**,(1983), 2 (in Russian).

4 A.P.Goncharov, Isomorphic classification of the spaces of infinitely differentiable functions, Ph.D.Thesis, Rostov State University, (1986) (in Russian).

5 A.Goncharov, A compact set without Markov's property but with an extension operator for C^∞ functions, *Studia Math.* **119**,(1996), 27-35.

6 A.Goncharov and M. Kocatepe , Isomorphic classification of the spaces of Whitney functions, *Michigan Math. J.*, **44**,(1997), 555-577.

7 A. Goncharov, Spaces of Whitney functions with basis, *Math.Nachr.* **220**,(2000), 45-57.

8 A.P.Goncharov and V.P.Zahariuta, Basis in the space of C^∞-functions on a graduated sharp cusp, submitted.

9 B.S. Mitiagin, Approximate dimension and bases in nuclear spaces, *Russian Math. Surveys,* **16**, 4 (1961), 59-127.

10 W. Pawłucki and W. Pleśniak, Markov's inequality and C^∞ functions on sets with polynomial cusps, *Math. Ann.*, **275**, (1986), 467-480.

11 W. Pawłucki, W. Pleśniak, Extension of C^∞ functions from sets with polynomial cusps, *Studia Math.* **88** (1988), 279-287.

12 W. Pleśniak, Markov's inequality and the existence of an extension operator for C^∞ functions, *J. Approx. Th.* **61** (1990), 106-117.

13 W. Pleśniak, Extension and polynomial approximation of ultradifferentiable functions in $\mathbb{R}^n$, *Bull.Soc.Roy.Sc. Liège* **63**, 5 (1994), 393-402.

14 J.Schmets, M.Valdivia, On the existence of continuous linear analytic extension maps for Whitney jets,*preprint Publ.Inst.Math. Univ.Liège* **95.011** (1995), 16 pages.

15 R.T. Seeley, Extension of C^∞ functions defined in a half space, *Proc. Amer. Math. Soc.*, **15**, (1964), 625-626.

16 E.M.Stein, Singular integrals and differentiability properties of functions, *Princeton Univ.Press*, 1970.

17 M. Tidten, Fortsetzungen von C^∞-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}^n$ definiert sind, *Manuscripta Math.* **27** (1979), 291-312.

18 D. Vogt, Charakterisierung der Unterräume von s , *Math.Z.* **155**, (1977), 109-117.

19 D. Vogt, Sequence space representations of spaces of test functions and distributions, *in: Functional Analysis, Holomorphy and Approximation Theory, G. I. Zapata (ed.), Lecture Notes in Pure and Appl. Math., Dekker* **83**, (1983), 405-443.

20 H.Whitney, Analytic extension of differentiable functions defined in closed sets, *Trans.Amer. Math.Soc.* **36** (1934), 63-89.

21 V.P. Zahariuta, Some linear topological invariants and isomorphisms of tensor products of scale's centers, *Izv. Sev. Kavkaz. Nauch. Centra Vysh. Skoly.*, **4**, (1974), 62-64 (in Russian).

22 A.Zeriahi, Inégalités de Markov et Développement en Série de Polynômes Orthogonaux des Fonctions C^∞ et A^∞, *in: Several Complex Variables: Proceedings of the Mittag-Leffler Institute, 1987-1988, ed.J.F. Fornaess, Math.Notes 38, Princeton Univ.Press, Princeton, New Jersey* (1993), 684-701.